Example 1: ($M_2(\mathbb{R})$) Let

$M_2(\mathbb{R})$ denote the set of all 2×2 matrices with real entries.

Let $A = [a_{i,j}]_{i,j=1}^{2}$,

$B = [b_{i,j}]_{i,j=1}^{2}$, and

$C = [c_{i,j}]_{i,j=1}^{2}$ be

in $M_2(\mathbb{R})$.

Define

$$\boxed{A+B = [a_{i,j}+b_{i,j}]_{i,j=1}^{2}}$$

(add the corresponding entries)

$A \cdot B$ is the 2×2 matrix

with

$$\boxed{(A \cdot B)_{i,j} = (a_{i,1} b_{1,j} + a_{i,2} b_{2,j})}$$

$\forall \ 1 \leq i,j \leq 2$.

Under these operations, $M_2(\mathbb{R})$ is a ring!

Check! Since $(\mathbb{R}, +)$ is a commutative group and addition is component-wise, $(M_2(\mathbb{R}), +)$ is a commutative group.

Let's check distributivity in one direction:

$$A(B+C) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{21}+c_{21} & b_{22}+c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}(b_{11}+c_{11}) + a_{12}(b_{21}+c_{21})) & (a_{11}(b_{12}+c_{12}) + a_{12}(b_{22}+c_{22})) \\ (a_{21}(b_{11}+c_{11}) + a_{22}(b_{21}+c_{21})) & (a_{21}(b_{12}+c_{12}) + a_{22}(b_{22}+c_{22})) \end{bmatrix}$$

$\downarrow$ multiplication distributes over addition in $\mathbb{R}$

$$= \begin{bmatrix} (a_{11}b_{11} + a_{11}c_{11} + a_{12}b_{21} + a_{12}c_{21}) & (a_{11}b_{12} + a_{11}c_{12} + a_{12}b_{22} + a_{12}c_{22}) \\ (a_{21}b_{11} + a_{21}c_{11} + a_{22}b_{21} + a_{22}c_{21}) & (a_{21}b_{12} + a_{21}c_{12} + a_{22}b_{22} + a_{22}c_{22}) \end{bmatrix}$$

$\Big\downarrow$ definition of "+" in $M_2(\mathbb{R})$

$$= \begin{bmatrix} (a_{11} b_{11} + a_{12} b_{21}) & (a_{11} b_{12} + a_{12} b_{22}) \\ (a_{21} b_{11} + a_{22} b_{21}) & (a_{21} b_{12} + a_{22} b_{22}) \end{bmatrix}$$

$$+ \begin{bmatrix} (a_{11} c_{11} + a_{12} c_{21}) & (a_{11} c_{12} + a_{12} c_{22}) \\ (a_{21} c_{11} + a_{22} c_{21}) & (a_{21} c_{12} + a_{22} c_{22}) \end{bmatrix}$$

$$= A \cdot B + A \cdot C$$

Associativity

Note that if $A \in M_2(\mathbb{R})$,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

A defines a (linear) function

$$S_A : \mathbb{R}^2 \to \mathbb{R}^2,$$

$$S_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}$$

for $x, y \in \mathbb{R}$

If $B \in M_2(\mathbb{R})$, then

$$\boxed{S_B \circ S_A = S_{BA}}.$$

Then associativity of matrix multiplication is equivalent to associativity of function composition, which we know to be true!

$$S_{C(BA)} = S_C \circ (S_{BA})$$

$$= S_C \circ (S_B \circ S_A)$$

associativity of function composition

$$= (S_C \circ S_B) \circ S_A$$

$$= (S_{CB}) \circ S_A$$

$$= S_{(CB)A}$$

So $S_{C\cdot(BA)} = S_{(CB)\cdot A}$

$\Rightarrow$ $C \cdot (BA) = (CB) \cdot A$

We need to prove this last implication.

Suppose $S_A = S_B$ where

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Then if $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$,

$$(S_A e_1) \cdot e_1$$

$$= \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \left(\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$= a_{11}.$$

Similarly,

$$(S_A e_2) \cdot e_2 = a_{22}$$

$$(S_A e_2) \cdot e_1 = a_{12}$$

$$(S_A e_1) \cdot e_2 = a_{21}$$

If $S_A = S_B$, then $\forall\, i, j$

$1 \leq i, j \leq 2$,

$S_A e_i = S_B e_i$, and so

$$(S_A e_i) \cdot e_j = (S_B e_i) \cdot e_j.$$

The left-hand side is $b_{j,i}$ and the right-hand side is $a_{j,i}$, so we get

$$b_{j,i} = a_{j,i} \quad \forall \ 1 \le i,j \le 2$$

$$\Rightarrow A = B.$$

It only remains to prove that

$$S_B \circ S_A = S_{BA}$$

We check this at the level of vectors.

$$(S_B \circ S_A)\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= S_B\left(S_A\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

$$= S_B\left(A \cdot \begin{bmatrix} x \\ y \end{bmatrix}\right)$$

$$= S_B\left(\begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}\right)$$

$$= \begin{bmatrix} b_{11}(a_{11}x+a_{12}y) + b_{12}(a_{21}x+a_{22}y) \\ b_{21}(a_{11}x+a_{12}y) + b_{22}(a_{21}x+a_{22}y) \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}a_{11}x + b_{11}a_{12}y + b_{12}a_{21}x + b_{12}a_{22}y \\ \\ b_{21}a_{11}x + b_{21}a_{12}y + b_{22}a_{21}x + b_{22}a_{22}y \end{bmatrix}$$

$$= \begin{bmatrix} (b_{11}a_{11} + b_{12}a_{21})x + (b_{11}a_{12} + b_{12}a_{22})y \\ \\ (b_{21}a_{11} + b_{22}a_{21})x + (b_{21}a_{12} + b_{22}a_{22})y \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= (B \cdot A) \begin{bmatrix} x \\ y \end{bmatrix} = S_{BA} \begin{bmatrix} x \\ y \end{bmatrix}$$

Recall: (unital and commutative rings)

A ring R is said to be commutative if $\forall\ x, y \in R$,

$$\boxed{x \cdot y = y \cdot x}$$

R is said to be unital if $\exists\ 1_R \in R$ such that

$$\boxed{1_R \cdot x = x \cdot 1_R = x}$$

$\forall\ x \in R$.

In the case that R is unital, $x \in R$ is said to be a unit if $\exists$ $y \in R$,

$$\boxed{x \cdot y = y \cdot x = 1_R}$$

Example 2: (a non-unital ring)

Let R be the even integers. All ring-theoretic properties, such as the commutative group structure, are inherited from $\mathbb{Z}$. The things we should check are that the operations are binary!

Let $x, y \in R$. Then

$\exists\, n, m \in \mathbb{Z}$,

$$x = 2n, \quad y = 2m.$$

Then $x+y = 2n+2m = 2(n+m) \in R$

$x \cdot y = (2n)(2m) = 2(2 \cdot n \cdot m) \in R$

So R will be a ring.

But there does not exist

$1_R \in R$, since if $z \in R$,

$z \cdot x = x \quad \forall\, x \in R$,

write $z = 2\ell$ for $\ell \in \mathbb{Z}$.

Then

$(2\ell) \cdot (2n) = 2n$

$$(2\ell - 1)(2n) = 0$$

$$\Rightarrow 2\ell - 1 = 0$$

Since we may choose $n = 1$.

$$\ell = 1/2 \,, \quad \notin \mathbb{R} .$$

So no such element can exist!

Definition: (ring homomorphism / isomorphism)

Let R, S be rings. A function $\varphi : R \to S$ is said to be a **ring homomorphism** if $\forall\ x, y \in R$,

1) $\underbrace{\varphi(x+y)}_{\text{addition in } R} = \underbrace{\varphi(x)+\varphi(y)}_{\text{addition in } S}$

and

2) $\underbrace{\varphi(x\cdot y)}_{\text{multiplication in } R} = \underbrace{\varphi(x)\cdot\varphi(y)}_{\text{multiplication in } S}$

If, in addition, φ is bijective, we say φ is a ring isomorphism and that R and S are isomorphic (as rings).

Definition: (field) A ring K is said to be a **field** if K is a commutative ring such that every $x \in K$ with x not equal to the additive identity is a unit.

Familiar Fields: the rational numbers $\mathbb{Q}$ (inverse of $\frac{a}{b}$ is $\frac{b}{a}$ for $a \neq 0$), the real numbers $\mathbb{R}$ (inverse of x is $\frac{1}{x}$ for $x \neq 0$) and the complex numbers $\mathbb{C}$ (inverse of $z \in \mathbb{C}$ is $\frac{1}{z}$ - but this requires complex conjugation to see that $\frac{1}{z} \in \mathbb{C}$!)